JOURNAL OF DIFFERENTIAL EQUATIONS 55, 1–29 (1984)

Kernels of Elliptic Operators: Bounds and Summability

DAVID GURARIE*

Department of Mathematics, Oregon State University, Corvallis, Oregon 97331

Received May 18, 1982; revised December 6, 1982 and April 18, 1983

In this paper we shall study the kernel of the resolvent $R = (\zeta - A)^{-1}$ and of some other related "functions of A" (e.g., e^{-tA}) for elliptic operators A on $\mathbb{R}^n$, or more generally, for perturbations of elliptic operators. It turns out that the resolvent (consequently all other related functions) are given by an integral kernel, which is bounded by a convolution with a radial decreasing L^1-function. This result has numerous applications: bounds for L^p-spectrum of A, closedness, semigroup generation, essential selfadjointness, summability etc. In [GK1] we studied this problem for perturbation of constant-coefficient elliptic operators and established the following bound on the kernel $R_\zeta(x, y)$ of $(\zeta - A)^{-1}$,

$$|R_\zeta(x, y)| \leqslant c(\zeta)\, \rho^{n/m} H(\rho^{1/m}\, |x - y|); \qquad \rho = |\zeta|, \tag{1}$$

where H is the radial L^1-function

$$H(z) = \begin{cases} |z|^{-s}; & |z| \leqslant 1 \\ |z|^{-t}; & |z| \geqslant 1 \end{cases} \qquad (s < n < t). \tag{2}$$

Though the class of perturbations considered in [GK1] was wide enough to include such examples, as Schrödinger operators with "Coulomb" and certain "magnetic" potentials (see [RS, Ch. 10]); $-\Delta + B(x) \cdot \nabla + V(x)$

$$V = \sum \frac{c_i}{|x_i|} + \sum_{i<j} \frac{c_{ij}}{|x_i - x_j|} \qquad (x_i \in \mathbb{R}^3),$$

the method of [GK1] was limited to operators with only constant-coefficient leading part.

The purpose of the present paper is to extend the results of [GK1] to operators with variable coefficients, in particular, operators on manifolds. By doing so we also improve the type of bound (2) and show that for differential

* Present address: Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106.

0022-0396/84 $3.00

operators A,[1] H has actually an exponential decay at ∞. Precisely, the radial function H in (1) can be taken to be

$$H(z) = \begin{cases} |z|^{-s}; & |z| \leqslant 1 \\ |z|^{-t} e^{-\gamma|z|}; & |z| > 1 \end{cases} \tag{3}$$

with $s < n$, $t > n$ and $\gamma = \gamma_0 |\sin \theta/m|$, where $\theta = \arg \zeta$, $m =$ order of A and the constant γ_0 depends on the leading symbol of A.

Notice that qualitatively estimate (3) is the best possible in the class of elliptic operators. Indeed, for the Laplacian $-\Delta$ on $\mathbb{R}^3$, the resolvent R_ζ is computed explicitly

$$R_\zeta(x, y) = \frac{1}{|x-y|} e^{i\sqrt{\zeta}\,|x-y|},$$

i.e., $\gamma = |\sin \theta/2|$.

The main result of the paper (Theorem 1) establishes estimate (1) with the bound H of type (3) for any uniformly elliptic operator A_0 on $\mathbb{R}^n$ with real principal symbol and the same class of perturbations B as in [GK1]. The method of the proof is further elaboration of the "perturbation-series" technique of [GK1]. In [GK1] this technique was used to expand the resolvent R of the operator $A = A_0 + B$ (A_0-elliptic leading part, B-perturbation) in terms of $R^0 = (\zeta - A_0)^{-1}$ and B

$$R = R^0 \sum_{k=0}^{\infty} (BR^0)^k. \tag{4}$$

For constant-coefficient A_0 each term of (4) is composed of multiplication operators with coefficients $b_\alpha(x)$ of B and convolution kernels $K_\alpha(x)$ of operators $D^\alpha(\zeta - A_0)^{-1}$. Each $K_\alpha(x)$ is the Fourier transform of a symbol (multiplier)

$$\sigma_\alpha(\xi) = \xi^\alpha(\zeta - a(\xi))^{-1}; \qquad \xi \in \mathbb{R}^n \tag{5}$$

$a(\xi)$ being the leading symbol of A (a homogeneous elliptic polynomial of degree m) and $|\alpha| < m$.

Multipliers $\sigma_\alpha(\xi)$ belong to certain negative order symbol classes, $\sigma_\alpha \in S_{1,0}^{-m+|\alpha|}$, in the standard terminology of pseudodifferential operators (see [Hö], [Ta]), which yields radial bounds for the kernel $K_\alpha(z)$

$$|K_\alpha(x-y)| \leqslant \text{Const} \begin{cases} |x-y|^{-n+m-|\alpha|}; & |x| \leqslant 1 \\ |x-y|^{-n-\delta}; & |x| \geqslant 1 \end{cases} \tag{6}$$

[1] The result of [GK1] were valid for more general pseudodifferential operators A.

with Const depending on so-called symbol class seminorms of σ_α. These bounds were crucial in the proof of Theorem 1 for constant-coefficient operators ([GK1, Theorem 2]).

Of course, in this approach it was important to know explicitly the complete symbol of the operator $D^\alpha(\zeta - A_0)^{-1}$.

However, in the variable-coefficient case we no longer know the complete symbol of $D^\alpha(\zeta - A_0)^{-1}$. Its "leading part" $\xi^\alpha/(\zeta - a(x, \xi))$ and any number of subsequent terms available by the standard pseudodifferential calculus (see, for instance, [Ta, Chap. III]) are not enough to bound the kernel K_α, since even the ∞-smoothing part can give a nonzero contribution to it.

However, it is possible to expand K_α in a geometric series, similar to (4), whose terms can be studied in the same fashion. This procedure, known as "freezing of coefficients" technique, is used to construct a parametrix of $\zeta - A$. To illustrate this procedure let us denote by $K = K(x, y)$ the kernel of the resolvent $(\zeta - A)^{-1}$ and by $K^0 = K^0(x, x-y)$ a ψDO with symbol $1/(\zeta - a(x, \xi)) \in S^{-m}_{1,0}$, the "principal part" of K. One easily verifies that

$$(\zeta - A)\, K^0 = \delta_{x-y} - L,$$

where L is a ψDO of some negative order, whose symbol is computed explicitly (see Section 1). Then

$$(\zeta - A)(K - K^0) = L; \qquad \text{i.e., } K - K^0 = KL$$

which gives an expansion

$$K = K^0 \sum_{k=0}^{\infty} L^k. \tag{7}$$

As with (4) we study (7) term by term, which amounts to estimating the products of ψDO's $\sigma(x, D)$ with symbols $\sigma(x, \xi) = b(x, \xi)/[\zeta - a(x, \xi)]^k$. Here $a(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\, \xi^\alpha$ is the leading symbol of A and $b(x, \xi)$ a polynomial in ξ of degree $<km$.

Now the exponential decay of the kernel $K_\sigma = \sigma(x; D)$ can be naturally explained in terms of its symbol. Notice that in general the rate of decay of the kernel (Fourier transform) at $\{\infty\}$ is equal to the degree of smoothness of its symbol (multiplier) at $\{0\}$ (cf. [GK1, Section 1]). However, to have an exponential decay σ needs to be holomorphic. Indeed, we show (Proposition 2) that the rational function $\sigma(x, \xi)$ extends to some tube domain $T = \{\xi + i\eta : |\eta| \leqslant \gamma\}$ in the complex space $\mathbb{C}^n$. Moreover for each η the real variable function $\sigma_\eta(\xi) = \sigma(\xi + i\eta)$ belongs to a negative order class $S^{-\delta}_{1,0}$, and the family $\{\sigma_\eta : |\eta| \leqslant \gamma\}$ is bounded in $S^{-\delta}_{1,0}$. The exponential decay easily follows now by moving the "contour" of integration ($\mathbb{R}^n$) into the complex space (see Section 1). Furthermore, the constant γ in the exponential gets a geometric interpretation as the radius of the maximal tube T about real space $\mathbb{R}^n$, which does not intersect the variety $\{\xi + i\eta : a(\xi + i\eta) = \zeta\}$. We estimate it from below by $\gamma_0 |\sin \theta/m|$.

Thus to prove Theorem 1 we use the same technique twice. First to analyse a geometric series (7), which represents the resolvent R^0 of the leading part A_0 of A in terms of leading part of the resolvent, K^0, and a ψDO L. Secondly, we use it to study a geometric series (4) which gives the resolvent of $A = A_0 + B$ in terms of R^0 and B.

Once Theorem 1 is established most of the corollaries and applications follow essentially [GK1, Sects. 2, 3]). In particular, we get

(a) closedness and a uniform bound on the spectrum of A in L^p-spaces,

(b) essential selfadjointness in L^2,

(c) a priori estimates,

(d) "resolvent summability," i.e., convergence $\zeta(\zeta - A)^{-1} f(x) \to f(x)$ in L^p and on the Lebesgue set of $f \in L^p$, as $\zeta \to \infty$ uniformly in any sector $\Omega_\theta = \{\zeta \in \mathbb{C} : |\arg \zeta| \geqslant \theta > 0\}$,

(e) existence of a strongly continuous holomorphic semigroup $\{e^{-tA}\}_{\mathrm{Re}\, t > 0}$ in L^p and C_0.

In Section 4 we discuss an extension of the above results to elliptic systems on $\mathbb{R}^n$. We consider elliptic systems with "real" principal symbol $a(x, \xi)$ but do not require $a(x, \xi)$ to be positive. This allows us to study higher order analogues of both "positive" Laplasians and Dirac-type operators with indefinite matrix $a(x, \xi)$. Of course the form of results changes for Dirac systems.

In conclusion let us make a few remarks concerning related results and the literature.

Kernels of the resolvent $(\zeta - A)^{-1}$, known as Green's functions, are useful in the spectral theory of Schrödinger operators and were studied by a number of authors (see [Ti], [Ka1] and references there). In [Ag] the resolvent kernel was a basic tool for derivation of spectral asymptotics of a general elliptic operator on compact domains.

The semigroup kernel $\{e^{-tA}\}$ ("heat" kernel) drew even more attention, due to its importance in spectral theory, geometry and recently topology. It has been extensively studied for Schrödinger operators and "Laplacians" on Riemannian manifolds.

In the works of Gårding [Ga], Donnelly [Do], and Cheng, Li, Yau [CLY] the bounds on the kernel $\{e^{t\Delta}\}$ (Δ—the Laplace operator) similar to that of the Euclidean "Gaussian" were obtained on a fairly general class of Riemannian manifolds. The recent work [CGT] by Cheeger, Gromov, Taylor extends there estimates to a wider class of "multipliers" $\phi(\sqrt{-\Delta})$ including the resolvent.

For various aspects of Schrödinger semigroup theory we refer to the recent survey of Simon [Si].

Let us notice that both "functions" $(\zeta - A)^{-1}$ and $\{e^{-tA}\}$ are related, and many properties of the one translate into appropriate properties of the other (see, for instance [HP]). However, for Laplacians and Schrödinger operators the study of semigroup usually precedes the study of resolvent. We go in the opposite direction, i.e., from resolvent to semigroup, as the former is better suited for the perturbation-series technique adopted in the paper.

Our result on essential selfadjointness extends some known facts in the theory of Schrödinger operators (see [RS], [Si], [Ka 2], [IK], etc.) to higher order elliptic operators with variable coefficients (see also [Br], [Co1], [Ch]). A recent paper [Du] by Dung deals with the same problem, but for a different class of perturbations.

Finally semigroup results (Theorem 3, Corollary 8) can be compared to an "abstract heat-diffusion" theory (see, for instance, [St]). Though our semigroups do not fall within the scope of "heat-diffusion" semigroups the results are shaper in two points:

(a) The semigroup $\{e^{tA}\}$ is shown to be holomorphic in the whole right half plane independant on L^p-class.

(b) The pointwise convergence $e^{-tA}f(x) \to f(x)$ is shown on the Lebesgue set of $f \in L^p$ rather than a.e. In fact, the maximal function of $\{e^{-tA}\}$ is dominated by the Hardy–Littlewood maximal function.

1. Radial Bounds for Operators $B(\zeta - A)^{-1}$

We shall need some definitions and results of the calculus of pseudodifferential operators (ψDO's) of classes $S^m_{1,0}$ $(-\infty < m < m)$ in the terminology of Hörmander [Hö]. Let us recall that a smooth function $a(x, \xi)$ $(x \in \mathbb{R}^n;\ \xi \in \mathbb{R}^n \backslash \{0\})$ belongs to $S^m_{1,0}$, if

$$\sup_{x,\xi} (1 + |\xi|)^{|\alpha| - m} |\partial^\beta_x D^\alpha_\xi a(x, \xi)| < \infty, \qquad \forall \alpha, \beta. \tag{1.1}$$

Here $\alpha = (\alpha_1 \cdots \alpha_n)$, $\beta = (\beta_1 \cdots \beta_n)$ denote multiindeces on $\mathbb{R}^n$ and $\partial^\beta_x, D^\alpha_\xi$ the corresponding partial derivatives in x and ξ with the standard convention that the differential $D = (1/i)\partial$. The right-hand side of (1.1) defines a seminorm on $S^m_{1,0}$, which we call $|a|_{\alpha,\beta}$.

Each symbol $a(x, \xi) \in S^m_{1,0}$ defines a pseudodifferential operator, ψDO,

$$Af(x) = a(x, D) f = \int e^{ix \cdot \xi} a(x, \xi) \hat{f}(\xi)\, d\xi, \tag{1.2}$$

where $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(\xi) = (2\pi)^{-n} \int f(x)\, e^{-ix\cdot\xi}\, dx.$$

Following [Co2], we introduce the class $CB^{\infty}(\mathbb{R}^n)$ of C^{∞}-functions on $\mathbb{R}^n$, which are bounded with all their partial derivatives,

$$\sup_{x\in\mathbb{R}^n} |(\partial^{\alpha} f)(x)| < \infty, \qquad \forall\alpha.$$

It is easy to see that multiplication with a function $a(x) \in CB^{\infty}$

$$f(x) \to a(x)\, f(x),$$

as well as convolution

$$f \to K * f = b(D)\, f,$$

with the kernel $K(x) = \hat{b}(x)$, $b \in CB^{\infty}$ define ψDO's in the class $S^0_{1,0}$, whose symbols are $a(x)$ and $b(\xi)$, respectively. Moreover, any differential operator $A = \sum_{|\alpha| \leqslant m} a_{\alpha}(x)\, D^{\alpha}$ with coefficients $a_{\alpha}(x) \in CB^{\infty}$ belongs to $S^m_{1,0}$. Also fractional powers of the Laplacian (Bessel potentials) $\Lambda^m = (1-\Delta)^{m/2}$, whose symbol is $a(\xi) = (1+|\xi|^2)^{m/2}$, belong to $S^m_{1,0}$ for all $m \in \mathbb{R}$. In fact, all ψDO's in $S^m_{1,0}$ $(-\infty < m < \infty)$ are generated in some sense by multiplications with functions $a \in CB^{\infty}$, partial differentiations D^{α} and Bessel potentials Λ^s $(-\infty < s < \infty)$ (see [Co2, Chaps. III, IV]).

A ψDO $a(x, D)$ in the class $S^m_{1,0}$ is given in general by a singular distribution kernel

$$K(x, x-y) = \int a(x, \xi)\, e^{i(x-y)\cdot\xi}\, d\xi. \tag{1.3}$$

However, for negative order operators (1.3) defines a "nice" integral kernel. Precisely,

Proposition 1. (see [Ne] and [GK1]). *The kernel $K(x, x-y)$ of an operator $a(x, D)$ in the class $S^{-m}_{1,0}$ $(m > 0)$ is differentiable in x and $z = x-y$ away from $\{0\}$ and admits the estimates*

$$|(\partial^{\alpha}_x K)(x, z)| \leqslant c \sum_{|\beta| \leqslant n+1} |a|_{\alpha,\beta} \begin{cases} 1 + |z|^{-n+m}; & (-\log|z|, \text{ if } n = m)\ |z| < 1 \\ |z|^{-t}; & |z| \geqslant 1. \end{cases} \tag{1.4}$$

Here $z = x-y$, constant c depends on n and m, and t is equal to the degree of smoothness of $a(x, \xi)$ in the variable ξ, at 0.

We call the radial function in the right hand side of (1.4) $H_{s,t}$, i.e.,

$$H_{s,t}(z) = \begin{cases} |z|^{-s}; & |z| \leqslant 1 \\ |z|^{-t}; & |z| > 1. \end{cases} \tag{1.5}$$

We also introduce a class of exponentially decaying radial functions

$$H_{s,t,\gamma}(z) = \begin{cases} |z|^{-s}; (-\log|z|); & |z| \leqslant 1 \\ |z|^{-t} e^{-\gamma|z|}; & |z| > 1. \end{cases} \tag{1.6}$$

Obviously $H_{s,t}$ with $s < n < t$ and $H_{s,t,\gamma}$ with $s < n, \gamma > 0$ belong to L^1. It is easy to find their L^p-classes as well as mixed $L^{p,q}$-classes with respect to the splitting of $\mathbb{R}^n$ into the sum of subspaces $\mathbb{R}^n = E \oplus F$.

We consider differential operators $A = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha$ with smooth coefficients $a_\alpha \in CB^\infty$ and real principal symbol $a(x, \xi)$. As usual ellipticity of A will mean that

$$a(x, \xi) \neq 0, \qquad \forall\, \xi \neq 0$$

or equivalently

$$a(x, \xi) \geqslant c\, |\xi|^m$$

with constant c depending on x.

We call A *uniformly elliptic* if the leading symbol satisfies

$$c_1 |\xi|^m \leqslant a(x, \xi) \leqslant c_2 |\xi|^m; \qquad \forall\, \xi \in \mathbb{R}^n \tag{1.7}$$

with constants $c_1, c_2 > 0$ independant of $x \in \mathbb{R}^n$. The order of an elliptic operator with real principal symbol must be even $m = 2k$.

We want to study the kernel of the resolvent $(\zeta - A)^{-1}$ of an elliptic differential operator $A = a(x, D)$ of order m and, more generally, the kernel of an operator $B(\zeta - A)^{-1}$, where the order of B is less than m. The following Lemma gives an estimate of this kernel.

LEMMA 1. *Let A be a uniformly elliptic homogeneous differential operator of order m with real positive symbol $a(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\, \xi^\alpha$ and $B = \sum_{|\alpha| = m'} b_\alpha(x) D^\alpha$ be any homogeneous differential operator of order $m' < m$. We assume that both A and B have coefficients in CB^∞.*

There exists a "parabola-shaped" domain Ω about positive real axis in $\mathbb{C}$ such that for each $\zeta = \rho e^{i\theta}$ in the complement of Ω the kernel $K = K_\zeta(x, y)$ of the operator $B(\zeta - A)^{-1}$ is bounded by L^1-dilations by a radial function $H_{s,t,\gamma}$ (1.6). *Precisely*

$$|K_\zeta(x, y)| \leqslant c_1(\zeta)\, \rho^{-1 + (m' + n)/m} H_{s,t,\gamma}(\rho^{1/m} |x - y|) \quad (\zeta = \rho e^{i\theta}) \tag{1.7}$$

where $s = n - (m - m')$, $t > 0$, $\gamma = \gamma_0 |\sin \theta/m|$. *The coefficient*

$$c_1(\zeta) = \frac{c_0(\theta, b)}{1 - c(\theta, a)\,\rho^{-1/m}} \tag{1.8}$$

with c_0 *and* c *depending on* $\theta = \arg \zeta$ *as* $O(|\theta|^{-\tau})$ *and symbols* $a(x, \xi)$ *and* $b(x, \xi)$.

The result of this type for constant-coefficient operators was a part of the main theorem in [GK1]. But the bound of [GK1] did not have exponential factor, which appears in the context of differential operators.

Proof. From the general theory [Co2] it follows that both $(\zeta - A)^{-1}$ and $B(\zeta - A)^{-1}$ are ψDO's of some negative order. However, a complete symbol of $B(\zeta - A)^{-1}$, which is needed in order to use Proposition 1, is not available unless A has constant coefficients.

To circumvent the difficulty we denote by $K^0 = K_\zeta^0(x, y)$ the kernel of a ψDO with symbol $1/(\zeta - a(x, \xi)) \in S_{1,0}^{-m}$, the parametrix of $\zeta - A$.

By Proposition 1, $K^0(x, z)$ is a smooth kernel in x and z away from $\{0\}$ and one can easily verify that

$$(\zeta - A)\,K^0 = \delta_{x-y} - L, \tag{1.9}$$

where L is a ψDO of a negative order, whose kernel

$$L(x, z) = \sum_{\alpha' < \alpha} a_\alpha(x) \binom{\alpha}{\alpha'} (D_x^{\alpha - \alpha'} D_z^{\alpha'} K^0)(x, z) \tag{1.10}$$

$\{a_\alpha\}$ being the coefficients of A, and whose symbol

$$\sigma_L(x, \xi) = \sum_{\alpha' < \alpha} \binom{\alpha}{\alpha'} a_\alpha(x)\, \xi^{\alpha'} D_x^{\alpha - \alpha'} \left(\frac{1}{\zeta - a}\right). \tag{1.11}$$

Let $K = K_\zeta(x, y)$ denote the kernel of the resolvent $(\zeta - A)^{-1}$. The formal relation between K and K^0 is obtained by writing

$$(\zeta - A)\,K = \delta_{x-y} \tag{1.12}$$

and subtracting (1.19) from (1.12). This yields

$$(\zeta - A)(K - K^0) = L \qquad \text{or} \qquad K = K^0 + KL.$$

Iterating the latter identity K expands into a geometric series

$$K = K^0(I + L + L^2 + \cdots). \tag{1.13}$$

We shall estimate each term K^0L^k of (1.13) to show that the kernel K admits radial bounds (1.7) and then verify the resolvent identity $(\zeta - A)K = I$.

First let us rewrite the symbol of L in a more convenient form. We use the following iterated chain rule for derivatives of composite functions,

$$D_x^\beta(\phi \circ a) = \sum c_{\beta^1\ldots\beta^k}\phi^{(k)} \circ a \prod_{j=1}^{k} D_x^{\beta^j} a, \qquad 1 \leqslant k \leqslant |\beta|, \beta^1 + \cdots + \beta^k = \beta, \tag{1.14}$$

the summation being taken over all partitions of β into the sum of multiindeces $\beta^1 \cdots \beta^k$ and $c_{\beta^1\ldots\beta^k}$ being certain universal combinatorial coefficients. Substituting (1.14) into (1.11) with $\phi(\lambda) = 1/(\zeta - \lambda)$ and rearranging terms, we can rewrite (1.11) as

$$\sigma_L(x, \xi) = \sum \phi_{\beta^1\ldots\beta^k} \frac{\xi^{\beta^0}}{(\zeta - a)^{k+1}} \prod_{j=1}^{k} D_x^{\beta^j}(a),$$
$$\beta^0 + \beta^1 + \cdots + \beta^k = \alpha, \; 1 \leqslant k \leqslant m. \tag{1.15}$$

Here $\phi_{\beta^1\ldots\beta^k}(x) \in CB^\infty$, the summation is over all partitions of α into the sum of multiindeces $\beta^0 + \cdots + \beta^k$ with $|\beta^0| = |\alpha'| < m$.

Let us introduce a notation $\bar{\beta}$ for a tuple of multiindeces $(\beta^0, \beta^1, \ldots, \beta^k)$ s.t. $\sum_0^k |\beta^j| = m$. Then (1.15) takes a form

$$\sigma_L(x, \xi) = \sum_{\bar{\beta}} \phi_{\bar{\beta}}(x)\, \sigma_{\bar{\beta}}(x, \xi), \tag{1.16}$$

where

$$\sigma_{\bar{\beta}}(x, \xi) = \frac{\xi^{\beta^0}}{(\zeta - a)^{k+1}} \prod_{j=1}^{k} (D_x^{\beta^j} a) \in S_{1,0}^{\delta} \quad (\delta = |\beta^0| - m \leqslant -1). \tag{1.17}$$

The next step will be to estimate the kernel $L_{\bar{\beta}}$ of the ψDO $\sigma_{\bar{\beta}}(x, \dot{D})$. First we "scale out" the absolute value of the complex parameter $\zeta = \rho e^{i\theta}$, using the homogeneity of $a(x, \xi)$ and its derivatives. On the symbol level we get

$$\begin{aligned}\sigma_{\bar{\beta}}(x, \xi) &= \rho^{-1+|\beta_0|/m} \frac{(\rho^{-1/m}\,\xi)^{\beta^0}}{[e^{i\theta} - a(x; \rho^{-1/m}\xi)]^{k+1}} \prod_{j=1}^{k} (D_x^{\beta^j} a)(x; \rho^{-1/m}\xi) \\ &= \rho^{-1+|\beta^0|/m}\, \sigma_{\bar{\beta},\theta}(x, \rho^{-1/m}\xi).\end{aligned}$$

The dilation in the ξ-variable of symbols results in the dual L^1-dilation of kernels, i.e.,

$$L_{\bar{\beta}}(x, z) = \rho^{-1+(|\beta^0|+n)/m} L_{\bar{\beta},\theta}(x; \rho^{1/m}z) \qquad (z = x - y).$$

If we were content with radial bounds of type (1.3) we could apply Proposition 1 directly to $\sigma_{\bar{\beta},\theta}$ as in [GK1]. However, to get an exponential decay we shall need analytic continuation of $\sigma_{\bar{\beta},\theta}$ in the complex space $\mathbb{C}^n$.

PROPOSITION 2. *With the above notations and assumptions there exists a constant $\gamma_0 > 0$, such that the symbol $\sigma = \sigma_{\bar{\beta},\theta}(x, \xi)$ extends as a holomorphic function in the variable $\xi + i\eta \in \mathbb{C}^n$ on the tube domain*

$$T = \{\xi + i\eta : |\eta| \leqslant \gamma_0 |\sin \theta/m|\}, \qquad (\theta = \arg \zeta).$$

Moreover, for each η in T, $\sigma_\eta(\xi) = \sigma(\xi + i\eta) \in S_{1,0}^{|\beta 0| - m}$ and its seminorms are bounded uniformly in η

$$|\sigma_\eta|_{\nu,\mu} \leqslant c_{\nu,\mu}(\theta), \qquad \forall_\eta,$$

where $c_{\nu,\mu}(\theta) = O(|\theta|^{-t})$ with $t = |\nu + \mu|$.

The argument of Proposition 2 is based upon the following estimate: let a, θ be positive numbers and m an even integer. Then for all $|r| < |\sin \theta/m| / \sqrt[m]{a}$ the parabolic set Ω in $\mathbb{C}$,

$$\Omega = \{a(t + ir)^m \mid t \in \mathbb{R}\}$$

does not contain $e^{i\theta}$. Moreover the distance between $e^{i\theta}$ and $a(t + ir)^m$ is estimated from below

$$|e^{i\theta} - a(t + ir)^m| \geqslant \begin{cases} m|\sin\theta/m - \sqrt[m]{a}\, r|; & \text{for small } t, |t| \leqslant \dfrac{2^{m+1}}{\sqrt[m]{a}}; \\ \dfrac{a}{2} t^m; & \text{for large } t, |t| > \dfrac{2^{m+1}}{\sqrt[m]{a}}. \end{cases} \tag{1.18}$$

To derive (1.18) for large t we write

$$|e^{i\theta} - a(t + ir)^m| = \Big|\underbrace{at^m + \cdots}_{\text{Re}} \ + \underbrace{iat^{m-1} + \cdots}_{\text{Im}} \ - e^{i\theta}\Big|$$

$$\geqslant a[t^m - t^{m-1}(1 + r/\sqrt[m]{a})^m].$$

Remembering that $r/\sqrt[m]{a} < \sin \theta/m$ we estimate the latter expression by $(a/2)\, t^m$ provided $t \geqslant 2^{m+1}$.

For small t two complex numbers $w_0 = e^{i\theta}$ and $w = \sqrt[m]{a}(t + ir)$ are close, so by Taylor's theorem

$$|w_0^m - w^m| \sim m\, |w_0|^{m-1}\, |w_0 - w|.$$

Hence

$$|e^{i\theta} - a(t+ir)^m| \geqslant m \inf_t |e^{i\theta/m} - \sqrt[m]{a}(t+ir)| = m\,|\sin\theta/m - r\sqrt[m]{a}|.$$

Returning to Proposition 2 we need to estimate seminorms

$$\sup |\xi|^{m+|\alpha|}\, |D^\alpha([e^{i\theta} - a(x, \xi + i\eta)]^{-1})|.$$

Let us do it for $\alpha = 0$ as the argument is similar for higher order derivatives. Observe that by ellipticity the range of the function $\{a_\eta(\xi) = a(x, \xi + i\eta)\colon \xi \in \mathbb{R}^n;\ |\eta| \leqslant r_0\}$ is included in the set

$$\{a(t+ir)^m\colon t \in \mathbb{R};\ |r| \leqslant r_0\}$$

for some $a > 0$ depending on symbol $a(x, \xi)$. Due to uniform ellipticity constant a can be chosen independantly of $x \in \mathbb{R}^n$.

Then we estimate

$$\sup_\xi \frac{1 + |\xi|^m}{|e^{i\theta} - a(x, \xi + i\eta)|} \leqslant \sup_t \frac{1 + t^m}{|e^{i\theta} - a(t+ir)^m|}$$

and by (1.18) the latter is bounded by

$$\max\left\{\frac{2}{a};\ \frac{2^m}{m|\sin\theta/m - \sqrt[m]{a}\,r|}\right\}. \tag{1.19}$$

Choosing r_0 "ε-close" to its maximal value, $r_0 = (1-\varepsilon)(\sin(\theta/m)/\sqrt[m]{a})$ we get the desired estimate of the seminorm

$$\sup_\xi \frac{|\xi|^m}{|e^{i\theta} - a(x, \xi + i\eta)|} \leqslant \frac{2^m}{m\,|\sin\theta/m|} = O(|\theta|^{-1}), \tag{1.20}$$

for all $|\eta| \leqslant r_0 = (\sin(\theta/m)/\sqrt[m]{a})(1-\varepsilon)$. The constant γ_0 of Proposition 2 is thus equal to $(1-\varepsilon)/\sqrt[m]{a}$. Let us also observe that for higher order seminorms $|\sigma|_{\nu,\mu}$ the factor $(\sin\theta/m)$ to the negative power $|\nu + \mu|$ will appear in the estimate. This completes the proof.

After Proposition 2 exponential decay of the kernel $L = L_\beta$ easily follows by changing the "contour of integration." Indeed, take a symbol $\sigma(x, \xi) \in S_{1,0}^{-m'}$ which extends holomorphically in the tube-domain $T = \{\xi + i\eta \mid |\eta| \leqslant \gamma\}$, s.t. $|\sigma_\eta|_{\nu,\mu} \leqslant c_{\nu,\mu}$, $\forall_{\nu,\mu}$; where $\sigma_\eta(\xi) = \sigma(\xi + i\eta)$, and denote by $L = L(x, x-y)$ the kernel of $\sigma(x, D)$. Then

$$\begin{aligned} L(x,z) &= \int e^{i\xi\cdot z}\sigma(x,\xi)\,d\xi = \int e^{i(\xi + i\eta)\cdot z}\sigma_\eta(x,\xi)\,d\xi \\ &= e^{-\eta\cdot z} L_\eta(x,z), \end{aligned}$$

where L_η is the kernel of $\sigma_\eta(x, D)$. By Proposition 1

$$|L_\eta(x, z)| \leqslant \left(\sum |\sigma_\eta|_{0,\mu}\right) H_{s,t}(|z|) \leqslant cH_{s,t}(|z|)$$

and since η can be chosen arbitrary subject to $|\eta| \leqslant \gamma$ we get for L

$$|L(x, z)| \leqslant ce^{-\gamma|z|} H_{s,t}(z).$$

In our case $L = L_{\bar{\beta},\theta}$, $\gamma = \gamma_0 |\sin \theta/m|$, according to Proposition 2, and c is a complicated expression involving the coefficients of $a(\xi)$, γ_0 and θ. By Proposition 2, $c(\theta) = O(|\theta|^{-\tau})$, $(\tau > 0)$.

Thus we get the desired estimate for $L_{\bar{\beta},\theta}$

$$|L_{\bar{\beta},\theta}(x, z)| \leqslant c_{\bar{\beta}}(\theta)\, H_{s,t,\gamma}(|z|) \tag{1.21}$$

with $s = n - m + |\beta^0|$; $t > 0$; $\gamma = \gamma_0 |\sin \theta/m|$.

Now we can proceed to the kth term in the series expansion of $B(\zeta - A)^{-1}$

$$B(\zeta - A)^{-1} = \sum_{k=0}^{\infty} (BK^0)\, L^k. \tag{1.22}$$

We recall that $L = \sum_{\bar{\beta}} \phi_{\bar{\beta}}(x)\, L_{\bar{\beta}}$ and

$$L_{\bar{\beta}} = \rho^{-1 + (|\beta^0| + n)/m}\, L_{\bar{\beta},\theta}(x; \rho^{1/m} z), \qquad \zeta = \rho e^{i\theta}.$$

Let us notice that the above argument given for $L_{\bar{\beta}}$ carries over verbatim to the kernel $L_0 = BK^0$. In particular,

$$L_0(x, z) = \rho^{-1 + (n + m')/m}\, L_{0,\theta}(x, \rho^{1/m} z)$$

and $L_{0,\theta}$ admits radial bounds (1.21) with the constant c, which depends now on both symbols $a(\xi)$ and $b(\xi)$.

The kth term in (1.22) consists of N^k kernels ($N =$ the number of summonds in (1.16)) of the type $M = L_0 \cdot \phi_1 L_1 \cdot \ldots \cdot \phi_k L_k$, where $\phi_j L_j$ is one of $\{\phi_{\bar{\beta}} L_{\bar{\beta}}\}$. We write M as a multiple integral

$$M(x, y) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} L_0(x, y_1)\, \phi_1(y_1)\, L_1(y_1; y_2)$$
$$\cdots \phi_k(y_k)\, L_k(y_k, y)\, dy_1 \cdots dy_k.$$

Pulling out L^∞-norms of $\{\phi_j\}$, estimating each $L_j(y_j; y_{j+1})$ by a convolution kernel $c_{\bar{\beta}j} H_{s_j,t,\gamma}(|y_{j-}y_{j+1}|)$ and taking into consideration dilating factor $\rho^{1/m}$, we arrive at the inequality

$$|M(x, y)| \leqslant \left(\sum_1^k c_j \|\phi_j\|_\infty\right) \rho^{d + n/m} (H_{s_0 t_0 \gamma} * \cdots * H_{s_k t_k \gamma})(\rho^{1/m} |x - y|). \tag{1.23}$$

The exponent d of ρ in (1.23) is equal to

$$d = -k + \frac{1}{m} \sum_{j=1}^{k} |\beta^{0,j}| - 1 + \frac{m'}{m},$$

Since all multiindices $|\beta^0| \leqslant m-1$, we estimate d as

$$d \leqslant \frac{k}{m} + \left(\frac{m'}{m} - 1\right).$$

Let us notice that the convolution of kernels of type (1.6) is bounded by the kernel of the same type. Precisely,

$$H_{s_1 t_1 \gamma_1} * H_{s_2 t_2 \gamma_2} \leqslant c_2 H_{st\gamma}, \tag{1.24}$$

where $s = \min(s_1, s_2)$; $t = \min(t_1, t_2)$; $\gamma = \min(\gamma_1, \gamma_2)$. By induction (1.24) extends to any number of convolutions. Hence we get in (1.23)

$$|M(x, y)| \leqslant \left(\sum_{j=1}^{k} c_j \|\phi_j\|_\infty\right) \rho^{-k/m - 1 + (n+m')m} H_{st\gamma}(\rho^{1/m} |x-y|)$$

with $s = \min\{n - m + m';\ n - m + |\beta^0|\} \leqslant n - m + m'$. Here c_j stands for $c_{\bar{\beta}j}$, ϕ_j for $\phi_{\bar{\beta}j}$, etc.

Remembering that the kth term of (1.22) is the sum of the N^k products $\{M\}$ its kernel is bounded by

$$(BK^0L^k)(x, y)| \leqslant c \left\{\sum_{\bar{\beta}} c_{\bar{\beta}} \|\phi_{\bar{\beta}}\|_\infty) \rho^{-1/m}\right\}^k \rho^{-1+(m'+n)/m} H_{s,t,\gamma}(\rho^{1/m} |x-y|). \tag{1.25}$$

Introducing the function

$$C(\theta) = c \sum_{\bar{\beta}} c_{\bar{\beta}}(\theta) \|\phi_{\bar{\beta}}\|_\infty \tag{1.26}$$

and summing up a geometric series of bounds $C(\theta)\rho^{-1/m}$ in (1.25) we get the result

$$|(BK)(x, y)| \leqslant \left\{\frac{c_0(\theta)}{1 - C(\theta)\rho^{-1/m}}\right\} \rho^{-1+(m'+n)/m} H_{s,t,\gamma}(\rho^{1/m} |x-y|).$$

Of course, in order that the geometric series of bounds converges we need

$$\frac{C(\theta)}{\rho^{1/m}} < 1.$$

This condition defines a complement of a parabola-shaped domain Ω about positive real axis. Indeed, by Proposition 2 all $c_\beta(\theta)$ and consequently $C(\theta)$ given by (1.26) are $O(|\theta|^{-\tau})$ for some $\tau > 0$.

Finally, let us show that the kernel K given by series (1.13) represents the resolvent of A. It suffices to check the identity

$$(\zeta - A)Kf = f \tag{1.27}$$

on a dense subspace, e.g., all Schwartz functions $f \in \mathscr{S}$. Notice that $\mathscr{S}$ is preserved by ΨDO's of the type $(1, 0)$, in particular, all terms $(\zeta - A)K^0L^kf$ $(f \in \mathscr{S})$ $(k = 0, 1,...)$ exist and belong to $\mathscr{S}$. By (1.9) we get

$$\sum_{k=0}^{\infty} (\zeta - A)K^0L^kf = (I - L)\sum_{k=0}^{\infty} L^kf = f, \qquad f \in \mathscr{S}.$$

To proceed from (1.27) to an appropriate Banach space of functions E (e.g., $E = L^p$; $\mathscr{L}_s^p$; $C^{n+\alpha}$, etc.) it suffices to check that the operator K is founded on E. This is obviously true for all L^p $(1 \leqslant p < \infty)$ and C_0, due to L^1-bound (1.7) and can also be verified for Sobolev spaces $\mathscr{L}_s^p$ $(s \in \mathbb{R}$, $1 \leqslant p < \infty)$. Indeed, the boundedness of K in $\mathscr{L}_s^p$ amounts to L^p-boundedness of the operators

$$\Lambda^s(K^0L^k)\Lambda^{-s}; \qquad k = 0, 1,...;$$

where $\Lambda = (1 - \Delta)^{1/2}$ is the Bessel potential. But the latter can be analysed in the same way, as we did for K^0L^k, using the properties of conjugation $K \to \Lambda^sK\Lambda^{-s}$ of ΨDO's (see [Co, Chap. 3, 4]). This argument yields estimates (1.25) for the kernel $\Lambda^sK\Lambda^{-s}$ with the constant C depending now on s.

Thus $K = K_\zeta(x, y)$ is identified with the resolvent kernel of A in the above function spaces.

The lemma is proved.

An immediate corollary of Lemma 1 is radial bounds for the resolvent $R = (\zeta - A)^{-1}$ itself,

$$|R_\zeta(x, y)| \leqslant C\rho^{n/m-1} \begin{cases} (\rho^{1/m}|x-y|)^{-n+m}; & |x-y| \leqslant 1 \\ e^{-\gamma_0|\operatorname{Im}^m\sqrt{\zeta}|\,|x-y|}; & |x-y| \geqslant 1 \end{cases}.$$

These bounds can be used to derive a variety of results concerning A: bounds on the L^p-spectrum, summability, semigroup generation, etc. However, all those extend to a more general class of operators, which are perturbations of A with lower order terms. Therefore, we shall discuss them in the subsequent sections.

Another consequence of Lemma 1 is

COROLLARY 1. *The domain of the operator A in all L^p-spaces $(1 < p < \infty)$ is the mth Sobolev space $\mathscr{L}^p_m$.*

It suffices to show that two operators $(\zeta - A)\Lambda^{-m}$ and $\Lambda^m(\zeta - A)^{-1}$ are L^p-bounded. But both of them are given by Calderon–Zygmund type kernels, which are L^p for $1 < p < \infty$ (see [SW]). Indeed the first of two, $(\zeta - A)\Lambda^{-m}$, is a ΨDO in the class $S^0_{1,0}$ whose symbol is computed explicitly, $(\zeta - a(x, \xi))/(1 + |\xi|^2)^{m/2}$, while the second one expands into series (1.13). The first term of the series, $\Lambda^m K^0$, is again in the class $S^0_{1,0}$, and hence a Calderon–Zygmund operator. All other terms $\Lambda^m K^0 L^k$ $(k \geqslant 1)$ have negative order and due to L^1-bound (1.5) are L^p for all $1 \leqslant p \leqslant \infty$.

2. Perturbations of Elliptic Operators

In this section we shall study perturbations of elliptic operators $A = A + B$ with lower order terms, which have "bad," possibly singular coefficients. Precisely, we let $B = \sum_{|\alpha| < m} b_\alpha(x)\, D^\alpha$ with $b_\alpha \in L^{r_\alpha} + L^\infty$ on $\mathbb{R}^n$. More generally, $\{b_\alpha\}$ can be defined on quotient spaces $V_\alpha = \mathbb{R}^n/U_\alpha$ (U_α is a linear subspace of $\mathbb{R}^n$) and be in $L^{r_\alpha} + L^\infty(V_\alpha)$. The latter case is of interest due to physically important examples of Hamiltonians with so called "multichannel" potentials, like Coulomb potential.

The main result of the section is the following

THEOREM 1. *Let $A = A_0 + B$ be the sum of a uniformly elliptic homogeneous differential operator A_0 with real positive symbol and a perturbation B of the above type, i.e., $B = \sum_{|\alpha| < m} b_\alpha(x)\, D^\alpha$ with $b_\alpha \in L^{r_\alpha} + L^\infty(V_\alpha)$, $V_\alpha = \mathbb{R}^n/U_\alpha$. We assume that*

$$d = \max_\alpha \left\{ \frac{n_\alpha}{r_\alpha} + |\alpha| \right\} < m \qquad (n_\alpha = \dim V_\alpha). \tag{2.1}$$

Then for all $\zeta = \rho e^{i\theta}$ in the complement of a parabolashaped domain about positive real axis there exists an integral kernel $R_\zeta = R_\zeta(x, y)$, which satisfies

(I) *$R_\zeta(\zeta - A) f = f$ for all f in the intersection of domains of A_0 and B in L^p, $f \in \mathscr{D}_p(A_0) \cap \mathscr{D}_p(B)$. Here $1 \leqslant p < \infty$.*

(II) *For $1 \leqslant p \leqslant \min\{r_\alpha\}$ we have in addition*

$$(\zeta - A)\, R_\zeta f = f; \qquad \forall f \in L^p,$$

i.e., R_ζ is the resolvent $(\zeta - A)^{-1}$ of A in L^p for all $1 \leqslant p \leqslant \min\{r_\alpha\}$.

(III) *The kernel* $R_\zeta(x, y)$ *admits the radial bound*

$$|R_\zeta(x, y)| \leqslant c(\rho, \theta)\, \rho^{n/m-1} H_{s,t,\gamma}(\rho^{1/m}|x-y|), \tag{2.2}$$

where $s = n - m$, $t > n$, $\gamma = \gamma_0|\sin\theta/m|$ *and the coefficient*

$$C(\rho, \theta) = c_1(\zeta)\, c_2\left(\sum_\alpha \|b_\alpha\|\right), \qquad \|b_\alpha\| = \|b_\alpha\|_{r_\alpha} + \|b_\alpha\|_\infty \tag{2.3}$$

with the constants $c_1(\zeta)$ *of Lemma* 1 *and* c_2 *of Lemma* 2 *below.*

In the proof of Theorem 1 we shall need the notion of p-convolution and mixed (p, q)-convolution, introduced in [GK1].

For a pair of functions f, g on $\mathbb{R}^n$ we define their *p-convolution* as

$$(f \mathop{*}_p g)(x) = \left(\int |f(x-y)\, g(y)|^p\, dy\right)^{1/p}.$$

To define the mixed (p, q)-convolution we decompose $\mathbb{R}^n$ into the direct sum of suspaces $\mathbb{R}^n = U \oplus V$ and denote the U- and V-components of $x \in \mathbb{R}^n$ by x' and x'', respectively. Then

$$f \mathop{*}_{p,q} g = \left(\int_U \left(\int_V |f(x'-y'; x''-y'')\, g(y', y'')|^p\, dy'\right)^{q/p} dy''\right)^{1/p}. \tag{2.4}$$

So $f \mathop{*}_{p,q} g$ is equal to the $L^{p,q}$-mixed norm of the function $F(y) = F(y', y'') = f(x-y)\, g(y)$ with respect to the splitting $\mathbb{R}^n = U \oplus V$. The p-convolution corresponds to the trivial splitting $\mathbb{R}^n = \mathbb{R}^n \oplus \{0\}$.

We are interested in (p, q)-convolutions of radial functions $H_{s,t,\gamma}$.

LEMMA 2 (cf. [GK1, Section 3]. *Let* $H_{s,t,\gamma}$; $H_{s',t',\gamma'}$ *be a pair of functions of the type* (1.6). *If* s, s' *satisfy*

$$\max\{s, s'\} < \frac{n'}{p} + \frac{n''}{q} \tag{2.5}$$

then the (p, q)*-mixed convolution of* $H_{s,t,\gamma}$ *and* $H_{s',t',\gamma'}$ *is bounded by* $c_2 H_{s'',t'',\gamma''}$, *where* $s'' = \min\{s, s'\}$, $t'' = \min\{t, t'\}$, $\gamma'' = \min\{\gamma, \gamma'\}$, *and the constant* c_2 *depends only on* n, n', n''.

In [GK1] we stated this lemma for radial functions $H_{s,t}$ of the type (1.5). Of course, the addition of the exponential term does not affect the argument.

Notice that condition (2.5) of Lemma 2 is equivalent to the finiteness of the (p, q)-mixed norms,

$$\|H_{s,t}\|_{p,q} < \infty \text{ and } \|H_{s',t'}\|_{p,q} < \infty.$$

Lemma 2 extends by induction to a sequence of mixed convolutions.

COROLLARY 2 (Cf. [GK1, Sect. 3]). *Let* $\mathbb{R}^n = V_i \oplus U_i$ $(i = 1,..., k)$ *be a sequence of partitions of* $\mathbb{R}^n$, $\dim V_i = n_i$, $\dim U_i = n_i'$. *If the sequences of real numbers* $s_0, s_1,..., s_k$; $t_0, t_1,..., t_k$; $p_1,..., p_k$; $q_1,..., q_k$ *satisfy*

$$t_i > 0; \qquad \max\{s_i; \min\{s_1;... s_{i-1}\}\} \leqslant \left(\frac{n_i}{p_i} + \frac{n_i'}{q_i}\right), \tag{2.6}$$

for $i = 1, 2,..., k$, *then the sequence of mixed convolutions of* $\{H_{s_i t_i \gamma_i}\}_i^k = 0$ *is estimated as*

$$(H_{s_0 t_0 \gamma_0} * H_{s_1 t_1 \gamma_1}) * \cdots * H_{s_k t_k \gamma_k} \leqslant c_2^k H_{s', t', \gamma'} \tag{2.7}$$

where $s' = \min\{s_0, s_1 \cdots s_k\}$, $t' = \min\{t_0,..., t_k\}$, $\gamma' = \min\{\gamma_0,..., \gamma_k\}$.

The ith $*$ *in* (2.7) *means a* (p_i, q_i)*-mixed convolution with respect to the* i*th splitting* $\mathbb{R}^n = V_i \oplus U_i$.

The *proof of Theorem* 1 will mostly follow [GK1], Theorem 2. We denote by R^0 the resolvent of A_0 and define R via the perturbation series

$$R = \sum_{k=0}^{\infty} R^0(BR^0)^k. \tag{2.8}$$

First we shall establish radial bounds for the kernel R (statement III). It suffices to do it for each term $L_k = R^0(BR^0)^k$ of (2.8). We denote by $K_0(x, y)$, $K_\alpha(x, y)$ the kernels of ψDO's $(\zeta - A_0)^{-1}$ and $D^\alpha(\zeta - A_0)^{-1}$. Obviously, L_k consists of combinations of integral operators K_α and multiplications with coefficients $\{b_\alpha(x)\}$, i.e.,

$$L_k = \sum L_{\alpha_1} \cdots \alpha_k, \tag{2.9}$$

where

$$L_{\alpha_1 \cdots \alpha_k}(f) = K_0(b_1(K_1 \cdots b_k(K_k(f)) \cdots).$$

For simplicity we write K_j for K_{α_j}, b_j for b_{α_j}, etc. The multi-indices $\alpha_1 \cdots \alpha_k$ in (2.9) vary over the set of all multi-indices which appear in B.

It suffices to estimate each term $L_{\alpha_1 \cdots \alpha_k}$ of L_k. We write the kernel $L_{\alpha_1 \cdots \alpha_k}(x, y)$ as a multiple integral

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K_0(x, z_1)\, b_1(z_1)\, K_1(z_1, z_2) \cdots b_k(z_k)\, K_k(z_k, y)\, dz_1 \cdots dz_k$$

and estimate each kernel $K_\alpha(x,y)$ by the radial function $H_\alpha(|x-y|)$ according to Lemma 1:

$$|K_\alpha(x,y)| \leqslant H_\alpha(|x-y|) = c_1(\zeta)\,\rho^{-1+(|\alpha|+n)/m} H_{s,r,\gamma}(\rho^{1/m}|x-y|) \quad (2.10)$$

with $s=n-m+|\alpha|$, $t>n$. Then we have

$$|L_{\alpha_1\cdots\alpha_k}(x,y)| \leqslant H_0 * (b_1 H_1(\cdots * b_k H_k)\cdots), \quad (2.11)$$

Decomposing b_α into the sum of L^r- and L^∞-terms $b'_\alpha + b''_\alpha$ ($b'_\alpha \in L^{r_\alpha}$; $b''_\alpha \in L^\infty$) we can always assume that $b_\alpha \in L^{r_\alpha}$ for some r_α satisfying the assumption of Theorem 1. If each b_α belongs to L^{r_α} on the whole space $\mathbb{R}^n$, we use multiple Hölder inequalities to estimate (2.11) by a sequence of p-convolutions (cf. [GK1, Sect. 3])

$$\left(\prod_1^k \|b_j\|\right)(\cdots(H_0 *_{p_1} H_1) *_{p_2} H_2)\cdots *_{p_k} H_k), \quad (2.12)$$

where $p_i = r_i/(r_i-1)$ ($i=1,2,\ldots,k$). As above, r_i means r_{α_i}, $H_i = H_{\alpha_i}$, etc.

If the coefficients b_α are defined on quotients $V_\alpha = \mathbb{R}^n/U_\alpha$ ($\dim V_\alpha = n_\alpha$), we apply Hölder inequality in the variables of b_α only, so that (2.12) becomes a sequence of mixed convolutions,

$$\left(\prod_1^k \|b_j\|\, r_j\right)(\cdots(H_0 * H_1) * H_2)\cdots * H_k, \quad (2.13)$$

the ith $*$ in (2.13) means the $(p_i, 1)$-mixed convolution with respect to the ith splitting.

Each function H_α represents a dilation of the radial bound $H_{s,t,\gamma}$; $H_\alpha(z) = c(\zeta)\, H_{s,t,\gamma}(\rho^{1/m}|z|)$. We observe the following rule for the (p,q)-convolutions of dilations: if $f_\varepsilon(x) = f(\varepsilon x)$, $g_\varepsilon(x) = g(\varepsilon x)$, then

$$f_\varepsilon *_{p,q} g_\varepsilon = \varepsilon^{-(n'/p+n''/q)}(f *_{p,q} g)_\varepsilon. \quad (2.14)$$

Here $\mathbb{R}^n = U \oplus V$, $\dim U = n'$, $\dim V = n''$ and the (p,q)-convolution is taken with respect to this splitting.

Combining (2.10), (2.13) and (2.14) we get

$$|L_{\alpha_1\cdots\alpha_k}(x,y)| \leqslant c_k(\zeta)\left(\prod_j \|b_j\|\right)(H_{s_0 t\gamma} * H_{s_1 t\gamma})\cdots * H_{s_k t\gamma}(\rho^{1/m}|x-y|) \quad (2.15)$$

with the constant

$$c_k(\zeta) = c_1(\zeta)^{k+1}\rho^{(d/m-1)k+n/m-1}. \quad (2.16)$$

The exponent $k(d/m-1)$ of ρ in the right hand side of (2.16) comes from the sum of exponentials in (2.10) and the "dilation" formula (2.14). Precisely, we have

$$\sum_{\alpha} \left\{ \frac{|\alpha|+n}{m} - 1 - \left(\frac{n_\alpha}{p_\alpha} + n - n_\alpha \right) \right\}$$

$$= \sum_{\alpha} \left\{ \left(|\alpha| + \frac{n_\alpha}{r_\alpha} \right) \Big/ m - 1 \right\} \leqslant (d/m-1)\, k. \qquad (2.17)$$

The summation in (2.17) is over all $\alpha \in \{\alpha_1; \alpha_2; \ldots \alpha_k\}$, $p_\alpha = r_\alpha/(r_\alpha - 1)$ and $d = \max(n_\alpha/r_\alpha + |\alpha|)$.

Now let us apply Corollary 2 of Lemma 2 to a sequence of mixed convolutions in (2.15). For this we have only to check condition (2.6) of Corollary 2. But $s_i = s_{\alpha_i} = n - m + |\alpha_i|$ according to (2.10) and $s_0 = n - m$. Hence $\min\{s_0; s_1; \ldots s_{i-1}\} = s_0$; $\max\{s_i; \min\{s_0, \ldots, s_{i-1}\}\} = s_i$ $(i \geqslant 1)$.

If $\mathbb{R}^n = V_i \oplus U_i$ denotes the α_ith splitting ($\dim V_\alpha = n_i = n_{\alpha_i}$), then (2.6) is equivalent to

$$n - m + |\alpha_i| < \frac{n_i}{p_i} + n - n_i,$$

which after the substitution $1/p_i = 1 - 1/r_i$ becomes $n_i/r_i + |\alpha_i| < m$ (the condition of the Theorem). Thus Corollary 2 applied to $\{H_{s_i t \gamma}\}_{i=0}^{k}$ yields

$$(H_{s_0 t \gamma} * H_{s_1 t \gamma}) * \cdots * H_{s_k t \gamma} \leqslant c_2^k H_{s t \gamma},$$

where $s = \min\{s_j\} = s_0$. Hence we have

$$|L_{\alpha_1 \cdots \alpha_k}(x, y)| \leqslant \left(\prod_j \|b_j\| \right) c_k(\zeta)\, c_2^k H_{s,t}(\rho^{1/m} |x-y|). \qquad (2.18)$$

From (2.18) and (2.9), we get

$$|L_k(x, y)| \leqslant \left(\sum_{\alpha} \|b_\alpha\| \right)^k c_k(\zeta)\, c_2^k H_{st\gamma}(\rho^{1/m} |x-y|).$$

We introduce a new constant $C(\zeta) = c_1(\zeta)\, c_2(\sum_\alpha \|b_\alpha\|)$. The geometric series of bounds of L_k,

$$\sum_{k=0}^{\infty} (C(\zeta)\, \rho^{d/m-1})^k,$$

converges absolutely if

$$C(\zeta)\, \rho^{d/m-1} < 1.$$

The latter condition gives the domain Ω of Theorem 1 contained in the resolvent set of A. Remembering, that by Lemma 1

$$c_1(\zeta)=\frac{c_0(\theta)}{1-c(\theta)\rho^{-(m-1)/m}},$$

where $c_0(\theta)$, $c(\theta)$ are $(|\theta|^{-\tau})$ $(\tau>0)$, and the exponential $d/m-1<0$, one easily verifies that $\Omega=\{\zeta|\,C(\zeta)\,\rho^{d/m-1}\geqslant 1\}$ is a parabolic region about positive real axis. For each $\zeta\in\Omega$, the kernel $R_\zeta(x,y)$ is estimated by the sum of the geometric series

$$|R_\zeta(x,y)|\leqslant C(\zeta)(1-C(\zeta)\,\rho^{d/m-1})^{-1}\,\rho^{n/m}H_{s_0ty}(\rho^{1/m}\,|x-y|).$$

This proves the third statement of the theorem.

Next we want to check identities (I) and (II). The first is straightforward

$$R_\zeta(\zeta-A)\,f=\sum_{k=0}^{\infty}(R^0B)^kR^0(\zeta-A_0)\,f-\sum_{k=0}^{\infty}(R^0B)^{k+1}\,f=f.$$

In the latter expression all terms make sense for all $f\in\mathscr{D}_p(A_0)\cap\mathscr{D}_p(B)$. The second relation, which is crucial for R_ζ to be resolvent, requires additional assumptions. Namely, to get II one needs a dense subspace $\mathscr{D}\subset L^p$, which is mapped into $\mathscr{D}_p(A_0)\cap\mathscr{D}_p(B)$ by all terms of the series (2.8)

$$R^0(BR^0)^k(\mathscr{D})\subseteq\mathscr{D}_p(A_0)\cap\mathscr{D}_p(B),\qquad \forall_k \tag{2.19}$$

Indeed, if (2.19) holds we write as above

$$\forall f\in\mathscr{D},\qquad (\zeta-A)\,R_\zeta f=\sum_{k=0}^{\infty}(\zeta-A_0)\,R^0(BR^0)^kf$$
$$-\sum_{k=0}^{\infty}(BR^0)^{k+1}f=f.$$

To analyse hypothesis (2.19) we rewrite it in the equivalent form

$$(BR^0)^k\mathscr{D}\subseteq L^p,\qquad k=1,2,... \tag{2.20}$$

for a dense subset $\mathscr{D}$ of L^p.

The latter holds for all $1\leqslant p\leqslant\min\{r_\alpha\}$, since in this case BR^0 is bounded in L^p by Lemma 3 below. If $p>\min\{r_\alpha\}$ (2.20) is no longer true in general, as it is easy to construct an example of an operator $T=BR^0$, whose subspace of smooth vectors $\mathscr{D}^\infty(T)=\bigcap_{k=1}^\infty\mathscr{D}(T^k)$ is not dense in L^p, even $\{0\}$ (see [GK1]).

Thus to complete Theorem 1 we need

LEMMA 3 (see [GK1, Lemma 1]). *If a pair of operators A_0 and B satisfy assumptions of Theorem* 1, *then the operator norm of* $B(\zeta - A_0)^{-1}$ *in all* L^p *spaces* $1 \leqslant p \leqslant \min\{r_\alpha\}$ *is estimated as*

$$\|B(\zeta - A_0)^{-1}\| \leqslant C(\theta)\, \rho^{d/m-1}; \qquad C(\theta) = O(|\theta|^{-\tau}). \tag{2.21}$$

In [GK1] this result was stated for constant-coefficient A_0, but the argument based on radial bounds (1.5) is the same for uniformly elliptic operators.

One of the corollaries of Lemma 3 is

COROLLARY 3 (a priori estimates). *If a pair of operators* A_0, B *satisfies assumptions of Theorem* 1, *then for any* $\varepsilon > 0$, *there exists* $\lambda = \lambda_\varepsilon > 0$ s.t.

$$\|Bf\|_p \leqslant \varepsilon\, \|A_0 f\|_p + \lambda_\varepsilon \|f\|_p, \qquad 1 \leqslant p \leqslant \min_\alpha \{r_\alpha\} \tag{2.22}$$

for all f *in the domain* $\mathscr{D}_p(A_0)$ *of* A_0 *in* L^p.

As we already mention (Corollary 1) the domain of A_0 in L^p $(1 < p < \infty)$ is $\mathscr{L}^p_m$, the mth Sobolev space.

By Lemma 3 the domain $\mathscr{D}_p(B) \supset \mathscr{D}_p(A_0)$ for all $1 \leqslant p \leqslant \min\{r_\alpha\}$, which implies

COROLLARY 4. *For all* $1 < p \leqslant \min\{r_\alpha\}$ *the* L^p*-domains of* A *and* A_0 *are equal and both are equal to the mth Sobolev space* $\mathscr{L}^p_m$.

In the Hilbert space setting a priory estimates (2.22) along with Kato–Rellich theorem ([Ka, Chap. 5]) can be used to establish essential selfadjointness of the perturbation $A = A_0 + B$. But the latter will also follow from semiboundedness.

3. APPLICATIONS OF THEOREM 1

Immediate corollaries of Theorem 1 are

COROLLARY 5. *The spectrum of the operator* $A = A_0 + B$ *in all* L^p $(1 \leqslant p \leqslant \min\{r_\alpha\})$ *is included in the set*

$$\Omega' = \{\zeta = \rho e^{i\theta} \mid \rho^{d/m-1} \geqslant C(\zeta)\} \tag{3.1}$$

a "parabola-shaped" domain about the positive real axis. The resolvent of A has maximum decrease in all nonzero directions,

$$\|R_\zeta\| \leqslant \frac{c(\theta)}{|\zeta|}, \qquad \forall\, \theta \neq 0.$$

Notice, that the "parabolic" shape of the spectrum is what generally could be expected of the above type perturbations, as even in simple examples, like $A = -\Delta + \eta \cdot \nabla$, with some fixed $\eta \in \mathbb{R}^n$, the spectrum of A is exactly the parabola $\{\zeta = x + iy \in \mathbb{C} \mid x \geqslant y^2/|\eta|^2\}$.

COROLLARY 6. *The minimal operator A on $C_0^\infty(\mathbb{R}^n)$ is closeable in all L^p $(1 \leqslant p \leqslant \min\{r_\alpha\})$ and in C_0, provided all $\{b_\alpha(x)\}$ are bounded, $b_\alpha(x) \in L^\infty$.*

COROLLARY 7. *A formally symmetric operator $A = A_0 + B$ satisfying the hypothesis of Theorem* 1 *and such that* $\min_\alpha\{r_\alpha\} \geqslant 2$ *is semibounded from below and hence essentially self-adjoint in L^2.*

Notice that formal symmetry requires additional smoothness assumptions on coefficients b_α of B, namely, each b_α must be $|\alpha|$-smooth, i.e., $b_\alpha \in \mathscr{L}_s^{r_\alpha} + \mathscr{L}_s^\infty$, $s = |\alpha|$. As we mentioned earlier Corollary 6 extends some known results concerning essential self-adjointness of second order operators (cf. [Ch, Du, Ka2]).

Another application of Theorem 1 is to resolvent summability.

THEOREM 2. *If an operator $A = A_0 + B$ satisfies the assumptions of Theorem* 1, *then*

$$\zeta(\zeta - A)^{-1} f \to f$$

as $\zeta \to \infty$ uniformly in each sector $\Omega_\theta = \{\zeta : |\arg \zeta| \geqslant \theta > 0\}$ in L^p-norm $(1 \leqslant p < \infty)$ and for all $f \in C_0$ in L^∞-norm. Furthermore

$$\zeta(\zeta - A)^{-1} f(x) \to f(x)$$

on the Lebesgue set of any $f \in L^p$ $(1 \leqslant p \leqslant \infty)$.

Proof. Notice that the family of integral kernels $\{\zeta R_\zeta(x, y)\}_\zeta$ of the operators $\{\zeta(\zeta - A)^{-1}\}$ is bounded by Theorem 1 by the family of L^1-dilations of a radial decreasing L^1-function $H_{s,t,\gamma}$ $(s < n,\ \gamma > 0,\ t > n)$. In fact as follows from (2.2) the family $\{\zeta R_\zeta(x, y) \mid \zeta = \rho e^{i\theta}\}$ is bounded uniformly in each sector Ω_θ $(\theta > 0)$ for $\rho > \rho_0(\theta)$.

To prove the theorem we can use the following modification of the well-known Fourier analysis result ([SW, Theorem 1.25]).

PROPOSITION 3. *Let a family of kernels* $\{K_\varepsilon(x,y)\}_\varepsilon$ *be bounded by* L^1-*dilations of an* L^1-*radial decreasing function* h, *i.e.*,

$$|K_\varepsilon(x,y)| \leqslant \varepsilon^{-n} h(\varepsilon^{-1}|x-y|).$$

Assume also that the family of functions

$$\tilde{K}_\varepsilon(x) = \int K_\varepsilon(x,y)\,dy \to 1, \tag{3.2}$$

as $\varepsilon \to 0$ *for all* $x \in \mathbb{R}^n$. *Then as* $\varepsilon \to 0$

$$(K_\varepsilon f)(x) = \int K_\varepsilon(x,y) f(y)\,dy \to f(x),$$

in L^p- *and* C_0- *norm* $(1 \leqslant p \leqslant \infty)$ *and also pointwise on the Lebesque set of any* $f \in L^p$.

In [SW] this result was proved for convolution kernels $K_\varepsilon(x,y) = (1/\varepsilon^n) K((x-y)/\varepsilon)$, but the same argument works for integral kernels as well.

Thus to complete the proof we have only to check assumption (3.2) for $\zeta R_\zeta(x,y)$.

As above we shall expand ζR and ζR^0 into geometric series

$$\zeta R = \zeta R^0 + \sum_{k=1}^{\infty} \zeta R^0 (BR^0)^k, \tag{3.3}$$

and

$$\zeta R^0 = \zeta K^0 + \sum_{k=1}^{\infty} \zeta K^0 L^k, \tag{3.4}$$

and use estimates of the kth term in both series obtained in Lemma 1 and Theorem 1 to show that

$$\int (\zeta R_\zeta - \zeta K^0)(x,y)(x,y)\,dy \to 0 \qquad \text{as} \quad \zeta \to 0$$

uniformly in each sector Ω_θ. Since

$$\int \zeta K^0(x,y)\,dy = \frac{\zeta}{\zeta - a(x,0)} = 1, \qquad \forall \zeta$$

this would imply the result via Proposition 3.

For the kth term of (3.3) we use the estimate (2.2) of Theorem 1

$$\left| \int \zeta [R^0 (BR^0)^k](x, y)\, dy \right| \leqslant (C/\rho^\tau)^k \, \|H_{s,t,\phi}\|_1;$$
$$k = 0, 1, \ldots; \; C = C(\theta); \; \tau = 1 - d/m > 0.$$

Hence the difference $\int (\zeta R_\zeta - \zeta R^0)(x, y)\, dy$ is bounded by the sum of a geometric series

$$\sum_{k=1}^{\infty} (C/\rho^\tau)^k \to 0 \qquad \text{as} \quad \rho \to \infty \quad \text{in} \quad \Omega_\theta.$$

On the other hand, the kth term of (3.4) is estimated by (1.23) of Lemma 1

$$\left| \int \zeta (K^0 L^k)(x, y)\, dy \right| \leqslant \left(\frac{c_1(\theta)}{\rho^{1/m}} \right)^k \|H_{s,t,\gamma}\|_1; \qquad k = 0, 1 \ldots.$$

Hence the difference $\int \zeta (R^0 - K^0)(x, y)\, dy$ is bounded by the sum

$$\sum_{k=1}^{\infty} (c_1(\theta)/\rho^{1/m})^k \to 0 \qquad \text{as} \quad \rho \to \infty \quad \text{in} \quad \Omega_\theta.$$

This completes the proof.

As in [GK1] we can use resolvent summability and Cauchy integration to obtain a variety of other "multipliers" $\phi(A)$ and "summation families" $\{\phi_\varepsilon(A)\}_\varepsilon$. Indeed, in order that the Cauchy integral

$$\frac{1}{2\pi i} \int_\Gamma \phi(\zeta)(\zeta - A)^{-1} d\zeta$$

defines an operator with radially bounded kernel it suffices that ϕ be integrable over Γ with respect to the measure

$$d\mu = C(\theta) \Big/ \left(1 - \frac{C(\theta)}{\rho^{1-d/m}} \right) d\zeta.$$

One example of such a family is the one-parameter semigroup $\{e^{-tA}\}$ generated by A. Here a contour Γ consists of two rays $\{\rho e^{\pm i\theta} | \rho > \rho_0\}$ and a finite arc $\{\rho_0 e^{i\psi} | \, |\psi| > \theta\}$, with arbitrarily small θ and sufficiently large ρ_0. Then we have $d\mu \sim d\rho/\rho$, and in order that the analytic function $\phi_t(\zeta) = e^{-t\zeta}$ lie in $L^1(\Gamma, d\mu)$, we need

$$|\theta + \arg t| < \pi/2. \tag{3.5}$$

Since θ can be chosen arbitrarily small, it follows that the operator e^{-tA} has a radially bounded kernel

$$M_t(x, y) = \frac{1}{2\pi i} \int_\Gamma e^{-t\zeta} R_\zeta(x, y)\, d\zeta.$$

Moreover, the family of kernels $\{M_t\}_t$ can be shown to satisfy assumptions of Proposition 3 uniformly in each sector $\Omega_\theta = \{|\arg t| \leqslant \theta < \pi/2\}$. Thus we get

THEOREM 3. *An operator A of Theorem 1 is a generator of an analytic semigroup $\{e^{-tA}\}$ in the right half plane* $\operatorname{Re} t > 0$ *in all L^p-spaces $(1 \leqslant p \leqslant \min\{r_\alpha\})$ and in C_0, provided all $b_\alpha \in L^\infty$. Furthermore for all $f \in L^p$ we have L^p- and Lebesgue convergence*

$$e^{-tA} f(x) \to f(x)$$

uniformly in each sector $\Omega_\theta = \{t\colon |\arg t| \leqslant \theta < \pi/2\}$.

Remark 1. A similar result holds for semigroups generated by fractional powers A^s $(s \in \mathbb{R})$, whenever the latter can be obtained via "Dunford functional calculus" (i.e., Cauchy integration of the resolvent), for instance, if the spectrum of A lies in the right half plane. The corresponding "multiplier" is $e^{-t\zeta^s}$ and condition (3.5) becomes

$$|\arg t + s\theta| < \pi/2$$

which can be always satisfied for sufficiently small θ.

The latter results in their turn can be used to study boundary behavior of solutions of certain elliptic boundary value problems (cf. [GK2]).

4. Elliiptic Systems

In this section we want to show how the results of preceding sections extend to elliptic systems on $\mathbb{R}^n$.

First we give a definition of an elliptic system. Let E be a complex vector space equipt with a Hermitian inner product $\langle .,. \rangle$, and let $M(E)$ denote the algebra of all linear transformations on E with the involution $a \to a^*$. We denote by SM a subspace of all symmetric (Hermitian) matrices in $M(E)$, $a^* = a$. We consider spaces of E-valued functions $C^\infty(E); L^p(E)$ etc., with the natural pairing

$$\langle f, g \rangle = \int \langle f(x); g(x) \rangle\, dx.$$

A differential operator on $C^\infty(\mathbb{R}^n; E)$ (differential system) is given by

$$A = \sum_{|\alpha| \leqslant m} a_\alpha(x)\, D^\alpha,$$

whose coefficients $\{a_\alpha(x)\}_\alpha$ are matrix-valued functions. The leading symbol of A is an $M(E)$-valued function $a(x, \xi) = \sum a_\alpha(x)\, \xi^\alpha$. In the usual way one defines the classes of matrix-valued symbols $S^m_{1,0}(M(E))$ and E-valued ψDO's.

We say that $a(x, \xi) \in S^m_{1,0}(M(E))$ is *elliptic* if

(1) for any $|\xi| > r_0$ the matrix $a(x, \xi)$ is invertable in $M(E)$,

(2) the inverse function $a(x, \xi)^{-1} \in S^{-m}_{1,0}$.

For homogeneous elliptic polynomials $a(x, \xi)$ it suffices that $a(x, \xi')$ be invertible for all ξ' on the unit sphere, $|\xi'| = 1$.

We shall consider elliptic systems with "real" (symmetric) principal symbol, $a(x, \xi) \in SM$, $\forall x, \xi$. Let $\{\lambda_j = \lambda_j(x, \xi')\}_{j=1,\ldots}$ denote the spectrum of a symmetric matrix $a(x, \xi')$ ($|\xi'| = 1$).

We call $a(x, \xi)$ *uniformly elliptic*, if the absolute value of the lowest eigenvalue is bounded from below

$$\min_j |\lambda_j(x, \xi')| \geqslant c_1 > 0,$$

and the highest from above

$$\max_j |\lambda_j(x, \xi')| \leqslant c_2 < \infty,$$

uniformly in ξ' and x.

Notice that we do not require $a(x, \xi)$ to be positive. In particular, systems of Dirac type: $a \cdot \nabla = \sum_1^n a_j \partial_j$; $a_j \in SM$ and the matrix $a \cdot \xi = \sum_1^n a_j \xi_j$ nondegenerate for all $\xi \neq 0$, are allowed.

Another examples of elliptic systems are Hodge Laplacians

$$\Delta_p = d_{p-1} d^*_{p-1} + d^*_p\, d_p, \qquad p = 1, 2, \ldots$$

on exterior p-forms on $\mathbb{R}^n$ with respect to an "almost-Euclidean" metric $(g^{ij}(x))^n_{ij=1}$. The latter means that the metric tensor $(g^{ij}(x))$ and its (covariant) derivatives are all uniformly (in x) bounded with respect to Euclidean metric.

Most of the results of Sections 2 and 3 carry over to elliptic systems with some major changes however. As we no longer require $a(x, \xi)$ to be a positive matrix, the operators A_0 and $A_0 + B$ may not be bounded from below. In fact A_0 and A are semibounded iff the leading symbol $a(x, \xi)$ is positive, which is the case of Hodge Laplacians. On the other hand, Dirac system have indefinite symbol and their spectrum sweeps the whole real line. As the result the region Ω in $\mathbb{C}$, which bounds the L^p-spectrum of A and in whose complement the resolvent $(\zeta - A)^{-1}$ is given by a "nice" radially bounded kernel, will consist of two "parabolic pieces," Ω_+ about $\mathbb{R}_+$, and Ω_- about $\mathbb{R}_-$. Of course, contour integration of the resolvent is not applicable to operators with unbounded spectrum, in particular, they do not generate semigroups e^{-tA}.

Let us indicate the modification in Sections 2 and 3, which are necessary in order to carry them over to elliptic systems.

1. In Lemma 1, construction of the parametrix K^0 and the operator L. The only difference here is that instead of scalar symbols $[\zeta - a(x, \xi)]^{-1}$ we use matrix-valued $[\zeta - a(x, \xi)]^{-1}$.

Computing derivatives $D^\alpha[(\zeta - a)^{-1}]$ by the iterated chain rule (1.14) will result in

$$D^\alpha[(\zeta - a)^{-1}] = \sum_{\alpha^1 + \cdots + \alpha^k = \alpha} (\zeta - a)^{-1}(D^{\alpha^1}a)(\zeta - a)^{-1} \cdots \times (\zeta - a)^{-1}(D^{\alpha^k}a)(\zeta - a)^{-1} \tag{4.1}$$

due to noncommutativity of matrix-valued terms. In fact, "scalar" formula (1.14) follows from (4.1) when similar terms are collected.

2. Analytic continuation of symbols. We observe that each eigenvalue $\lambda_j(x, \xi)$ of the matrix $a(x, \xi)$ is a holomorphic function of ξ, homogeneous of degree m, which preserves sign, $+$ or $-$, on the real space $\mathbb{R}^n$. Then Proposition 2 applies to each λ_j to show that $[e^{i\theta} - a(x, \xi)]^{-1}$ extends as a holomorphic M-valued function into the tube $\{\xi + i\eta \,|\, |\eta| \leqslant \gamma_0 |\sin \theta/m|\}$ and satisfies estimates of Proposition 2.

The rest of the proof proceeds as in the scalar case, the only difference being the presence of both positive $\lambda_j(x, \xi) > 0$ and negative $\lambda_j(x, \xi) < 0$ terms.

The formulation of Lemma 1 will slightly change as it holds now only for ζ outside the union of two parabolas $\Omega_+ \supseteq \mathbb{R}_+$ and $\Omega_- \supseteq \mathbb{R}_-$. Moreover, in estimate (1.8) coefficients $c(\theta)$ and $c_1(\theta)$ will behave differently, both will be $O(|\theta(\pi - \theta)|^{-\tau})$.

Similarly affected is the formulation and argument of Theorem 1, but otherwise it remains the same.

We can state now the main results for elliptic systems.

THEOREM 4. *Let $A_0 = a(x, D)$ be a uniformly elliptic system and a* *perturbation* $B = \sum b_\alpha(x) D^\alpha$ *has matrix coefficients* $b_\alpha \in (L^{r_\alpha} + L^\infty)(M(E))$ *satisfying assumptions of Theorem* 1

$$\frac{n_\alpha}{r_\alpha} + |\alpha| < m, \qquad \forall\, \alpha.$$

Then for all ζ *in the complement of a parabolic region* Ω *about* $\mathbb{R}_+$ *(or the union of two parabolic regions* $\Omega_+ \supset \mathbb{R}_+$, $\Omega_- \supset \mathbb{R}_-$ *in the case of indefinite symbol* $a(x, \xi)$*) the resolvent* $(\zeta - A)^{-1}$ *of* $A = A_0 + B$ *is given by an integral* $M(E)$*-valued kernel* $R_\zeta(x, y)$, *which admit the radial bound*

$$\|R_\zeta(x, y)\| \leqslant c(\zeta)\, \rho^{-1+n/m} H(\rho^{1/m} |x-y|), \qquad \zeta = \rho e^{i\theta}$$

with the radial function H *of* (1.6).

Let us mention some consequences of Theorem 4. Corollaries 3–6 remain true for all elliptic systems. In particular, essential selfadjointness (Corollary 6) follows from an a priori estimate (Corollary 3) and the Kato–Rellich theorem.

Resolvent summability (Theorem 2) holds now in regions $\{\varepsilon < |\arg \zeta| < \pi - \varepsilon\}$.

However, for elliptic systems with positive leading symbol $a(x, \xi)$ all the above results have the same form as in the scalar case. In particular Cauchy integration is available to show that any elliptic system A with positive leading symbol generates a strongly continuous holomorphic semigroup $\{e^{-tA}\}_{\mathrm{Re}\, t>0}$ in $L^p(E)$-spaces and in $C_0(E)$.

Remark. All the above results extend in the obvious way to elliptic operators on asymptotically Euclidean manifolds X and elliptic systems on vector bundles over X. The quantities involved will depend now on the Riemannian distance and volume. But these are "almost" preserved by a diffeomorphism $\varphi: \mathbb{R}^n \to X$.

However, an extention of these results to "genuine" Riemannian manifolds will require a different approach (cf. [CGT]).

Note added in proof. After the paper was submited the author learned that exponential estimates of the "heat" kernel e^{-tA} were obtained earlier by S. D. Eidelman (*Math. Sbornik*, **33**, No. 3 (1953) and **38**, No. 1 (1956)). Semigroup estimate can be translated as usual into a suitable estimate of the resolvent kernel, and I. M. Gelfand and G. E. Shilov used the latter in order to study a "generalized eigenfunction expansion" of uniformly elliptic operators ("Generalized Functions", Vol. 3). Our method is different from that of Eidelman, as it is based on resolvent from which all other kernels, including semigroup, are constructed. An advantage of this approach is that it allows to use effectively perturbation technique and treat operators with nonregular coefficients.

REFERENCES

Ag. S. AGMON, "Lectures on Elliptic Boundary Value Problems," Van Nostrand, New York, 1964.

Br. F. E. BROWDER, Functional analysis and partial differential equations, II, *Math. Ann.* **145** (1962), 81–226.

CGT. J. CHEEGER, M. GROMOV, AND M. TAYLOR, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, preprint, 1982.

CLY. S. Y. CHENG, P. LI, AND S. T. YAU, On the upper estimate of the heat kernel of a complete Riemannian manifold, preprint.

Ch. P. R. CHERNOFF, Essential selfadjointness of powers of generators of hyperbolic equations. *J. Funct. Anal.* **12** (1973), 401–414.

Co1. H. O. CORDES, Self-adjointness of powers of elliptic operators on non-compact manifolds, *Math. Ann.* **195** (1972), 257–272.

Co2. H. O. CORDES, "Elliptic Pseudodifferential Operators, An Abstract Theory," Lecture Notes in Mathematics No. 756, Springer-Verlag, New York, 1979.

Du. N. X. DUNG, Essential selfadjointness for even order elliptic operators, preprint, 1982.

Do. H. DONELLY, Spectral geometry for certain non-compact Riemannian manifolds, *Math. Z.* **169** (1979), 63–76.

Gå. L. GÅRDING, Vectors, analytiques dans les representations de groupes de Lie, *Bull. Soc. Math. France* **188** (1960).

GK1. D. GURARIE AND M. KON, Radial bounds for perturbations of elliptic operators, *J. Funct. Anal.* **23** (1984), 99–123.

GK2. D. GURARIE AND M. KON, Summability and bounds for multipliers of elliptic operators on $\mathbb{R}^n$, *in* "Conference on Harmonic Analysis in Honor of A. Zygmund," pp. 607–619.

Hö. L. HÖRMANDER, Pseudo-differential operators, *Comm. Pure Appl. Math.* **18** (1965), 501–517.

HP. E. HILLE AND R. PHILLIPS, "Functional Analysis and Semigroups," Colloq. Publ. Vol. 31, Amer. Math. Soc., Providence, R.I., 1957.

IK. T. IKEBE AND T. KATO, Uniqueness of the selfadjoint extension of singular elliptic differential operators, *Arch. Rat. Mech. Anal.* **9** (1962), 77–92.

Ka1. T. KATO, "Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1975.

Ka2. T. KATO, Remarks on the selfadjointness and related problems for differential operators, *in* "Spectral Theory of Differential Operators" (I. W. Knowles and R. T. Lewis, Eds.), pp. 253–266, North–Holland, Amsterdam, 1981.

Ne. U. NERI, The integrable kernels of certain pseudodifferential operators, *Math. Ann.* **186** (1970), 155–162.

RS. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics, II, III, IV," Academic Press, New York, 1975.

Si. B. SIMON, Schrödinger semigroups, *Bull. Amer. Math. Soc.* **7** (1982), 447–526.

SW. E. STEIN AND G. WEISS, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, N.J., 1971.

St. E. STEIN, "Topics in Harmonic Analysis Related to the Littlewood–Paley Theory," Princeton Univ. Press, Princeton, N.J., 1970.

Ta. M. TAYLOR, "Pseudodifferential Operators," Princeton Univ. Press, Princeton, N.J., 1981.

Ti. E. C. TITCHMARSH, "Eigenfunction Expansions, II," Oxford Univ. Press, Oxford, 1958.